

SINGULARITY OF RANDOM MATRICES OVER FINITE FIELDS

KENNETH MAPLES

ABSTRACT. Let A be an $n \times n$ random matrix with iid entries over a finite field of order q . Suppose that the entries do not take values in any additive coset of the field with probability greater than $1 - \alpha$ for some fixed $0 < \alpha < 1$. We show that the singularity probability converges to the uniform limit with error bounded by $O(e^{-c\alpha n})$, where the implied constant and $c > 0$ are absolute. We also show that the determinant of A assumes each non-zero value with probability $q^{-1} \prod_{k=2}^{\infty} (1 - q^{-k}) + O(e^{-c\alpha n})$, where the constants are absolute.

1. INTRODUCTION

Let $q = p^f$ be a prime power and let \mathbb{F}_q be the finite field with q elements. Let ξ be a random variable that takes values in \mathbb{F}_q with probability distribution μ . We need μ to be suitably non-degenerate.

We say that μ is α -dense for some $0 < \alpha < 1$ if for every additive subgroup $T \triangleleft \mathbb{F}_q$ and $s \in \mathbb{F}_q$,

$$\mathbb{P}(\xi \in s + T) \leq 1 - \alpha.$$

Let A be an $n \times n$ random matrix whose entries are iid copies of ξ . We are interested in computing the typical spectral properties of A ; in particular, we would like to know how often the matrix is singular. As it turns out, as long as A is “dense” in the sense that its entries take values from an α -dense distribution, the singularity probability converges rapidly to the expected value.

Theorem 1.1 (Charlap, Rees, Robbins [2]). *With A and \mathbb{F}_q as above, we have*

$$\mathbb{P}(A \text{ is non-singular}) = \prod_{k=1}^{\infty} (1 - q^{-k}) + o_{q,\mu}(1).$$

Note that

$$\prod_{k=1}^n (1 - q^{-k}) = \frac{|\mathrm{GL}_n(\mathbb{F}_q)|}{|\mathrm{M}_n(\mathbb{F}_q)|}$$

is the density of invertible matrices in the set of all $n \times n$ matrices over \mathbb{F}_q .

In this article, we improve the error to an exponential decay depending only on the α -density of the distribution.

Theorem 1.2. *Let \mathbb{F}_q with $q = p^j$ and suppose $A \in M(n, \mathbb{F}_q)$ is a random matrix with iid entries which take values from an α -dense probability distribution. Then*

1991 *Mathematics Subject Classification.* Primary 15A52; Secondary 15A33, 60C05.

The author is supported by a Graduate Research Fellowship from the National Science Foundation.

we have the estimate

$$\mathbb{P}(A \text{ is non-singular}) = \prod_{k=1}^{\infty} (1 - q^{-k}) + O(e^{-c\alpha n})$$

where the implied constant and $c > 0$ are absolute.

The dependence on α is optimum: if we take μ such that $\mathbb{P}(\xi = 0) > 1 - c \frac{\log n}{n}$ for c suitably small, then we expect A to have a positive proportion of zero columns.

The techniques used to prove Theorem 1.2 also generalize to control the distribution of the determinant.

Theorem 1.3. *For all non-zero $t \in \mathbb{F}_q$, we have the formula*

$$\mathbb{P}(\det A = t) = q^{-1} \prod_{k=2}^{\infty} (1 - q^{-k}) + O(e^{-c\alpha n}).$$

where $c > 0$ and the implied constant are absolute.

We will consider more refined estimates for the singularity of random matrices in [10]. In particular we will prove estimates for the rank of non-degenerate matrices over a finite field and for matrices over other arithmetic rings.

The singularity of random matrices with continuous distributions over \mathbb{C} is well-understood. If the last $n - 1$ columns of an $n \times n$ matrix are linearly independent, they span a hypersurface in \mathbb{C}^n . Hypersurfaces have Lebesgue measure zero, so as long as the probability distribution of X is absolutely continuous the matrix is almost surely non-singular.

In contrast, matrices whose entries take values in a discrete distribution may be singular with strictly positive probability. For example, if the probability distribution μ is the Bernoulli distribution then the probability that the first two columns are linearly dependent is 2^{1-n} . Komlós was the first to show in [7] that the singularity probability of an $n \times n$ Bernoulli matrix over the integers converges to zero. He subsequently generalized this proof to general distributions in [8].

It was significantly more difficult to improve this bound to an exponential rate. For Bernoulli matrices, the first proof was found by Kahn, Komlós, and Szemerédi, who found the bound c^n for some $c < 1$ in [6]. Tao and Vu improved the bound to $(3/4)^n$ in [13] and [14], and it was further improved to $(1/\sqrt{2})^n$ in [1] by Bourgain, Vu, and Wood.

These estimates on the singularity probability are derived from progress on the Littlewood-Offord problem. The classical Littlewood-Offord problem asks, for a fixed $(a_1, \dots, a_n) \in \mathbb{R}^n$, how many of the signed sums $\pm a_1 \pm \dots \pm a_n$ can lie in a given interval? The problem is so called after Littlewood and Offord's analysis of the real zeros of random polynomials in [9]. Erdős improved the inequality in [3] to show that for $|a_k| \geq 1$ the number of sums in the interval is bounded above by $O(n^{-1/2} 2^n)$, and that this inequality is sharp for the all-ones vector. In general, Littlewood-Offord inequalities give a correspondence between structure in the coefficients of the vector (a_1, \dots, a_n) and bounds on the number of signed sums in a given interval.

In this paper we prove three Littlewood-Offord theorems for finite fields. We have replaced the random sums from the classical problem with inner products $w \cdot X$, where $w \in \mathbb{F}_q^n$ is a fixed vector of (mostly) non-zero residues and X is a random vector with iid entries taken from an α -dense probability distribution. The

techniques we employ go back to Halász in [4], who showed that the probability $\mathbb{P}(X \cdot w = 0)$ can be controlled by finding additive structure in the level sets of the Fourier transform of $1_{X \cdot w = 0}$. We also crucially rely on arguments developed in [6] and [13], where it was discovered that $\mathbb{P}(X \cdot w = 0)$ can be controlled by constructing auxiliary random vectors with larger concentration probabilities. We will discuss this idea further in Section 2.3.

The key new advance to study matrices over finite fields is an inverse theorem for random sums $w \cdot X$ which are almost uniformly distributed, but differ from the uniform distribution by an exponentially small quantity. In this setting we show that the coefficients of w must lie in a small subset $R \subseteq \mathbb{F}_q^n$. Because the sums are almost uniformly distributed, we can enumerate all such vectors.

The requirement that μ be an α -dense probability distribution can be weakened. As discussed in [2], let $\theta \in \mathbb{F}_8$ be a primitive element and consider the uniform distribution μ on $\{0, 1, \theta, 1 + \theta\}$. With the methods from this article is easy to show that we recover the expected singularity probability $\prod_{k=1}^{\infty} (1 - 8^{-k}) + O(e^{-cn})$ while μ is supported on an additive subgroup of \mathbb{F}_2^3 . This was first done by Kahn and Komlós in [5] where they showed that it suffices to assume that μ does not concentrate on affine *subfields*; i.e. subsets of the form $\beta\mathbb{F}_{p^d} + \gamma$ for $d \mid f$ and $\beta, \gamma \in \mathbb{F}_q$.

With this weaker condition we can construct examples that do not have an exponentially small error term. For example, let f be a large prime, $\theta \in \mathbb{F}_{p^f}$ a primitive element and μ uniformly distributed on $\{0, 1, \theta, 1 + \theta\}$. After expanding the determinant we see that $\det A$ can only take values in the additive subgroup $\langle 1, \theta, \theta^2, \dots, \theta^n \rangle$. This appears to be the only obstruction, and such probability distributions will be considered in a forthcoming paper.

In Section 2 we prove Theorem 1.2 for dense distributions over general finite fields. In Section 3 we prove the Littlewood-Offord theorems we need to establish Theorem 1.2. Finally, we prove Theorem 1.3 in Section 4.

2. PROOF OF THEOREM 1.2

Let X_1, \dots, X_n denote the columns of A . For convenience we will let $X \in \mathbb{F}_q^n$ denote an independent random vector with iid entries distributed according to μ ; thus each X_ℓ is an iid copy of X for $1 \leq \ell \leq n$.

We expose each column X_k in turn, from X_n to X_1 , and check whether it lies in the span of the previously exposed columns. Let $W_k := \langle X_{k+1}, \dots, X_n \rangle$ denote the span of the final $n - k$ columns of A . By conditional expectation,

$$\mathbb{P}(A \text{ is non-singular}) = \prod_{k=1}^n \mathbb{P}(X_k \notin W_k \mid \text{codim } W_k = k)$$

Suppose that V is a deterministic subspace of \mathbb{F}_q^n of codimension k . If X_k were chosen *uniformly* from \mathbb{F}_q^n then we would have $\mathbb{P}(X_k \notin V) = 1 - q^{-k}$ and the theorem would follow. In fact, we can show that for sufficiently small k this equality holds with exponentially small error.

Proposition 2.1. *There is an absolute constant $\eta > 0$ such that, for all $1 \leq k \leq \eta n$, we have the estimate*

$$\mathbb{P}(X_k \in W_k \mid \text{codim } W_k = k) = q^{-k} + O(e^{-cn})$$

where the implied constant and $c > 0$ are absolute.

For columns X_k with $k > \eta n$ we have $q^{-k} = O(e^{-cn})$, so it suffices to show that

$$\mathbb{P}(X_k \in W_k \mid \text{codim } W_k = k) = O(e^{-c\alpha n}).$$

This is guaranteed by the following lemma, first recorded in [11].

Lemma 2.2 (Odlyzko). *For any fixed subspace V of \mathbb{F}_q^n and random vector $X \in \mathbb{F}_q^n$ that is α -dense, we have the bound*

$$\mathbb{P}(X \in V) \leq (1 - \alpha)^{\text{codim } V}.$$

Proof of Lemma 2.2. Let k denote the codimension of V . We can find $n - k$ coordinates $\tau \subseteq [n]$ such that V is a graph over τ . If we condition on the coordinates of X in τ , then there is a unique choice for the remaining coordinates $[n] \setminus \tau$ for $X \in V$. Since μ is α -dense, the probability that each entry of X assumes the required value is bounded by $1 - \alpha$, and the result follows from the independence of the entries. \square

We will now prove Proposition 2.1. It is convenient to distinguish four kinds of subspaces that W_k can represent as X_{k+1}, \dots, X_n vary. Fix absolute constants δ , d , and D ; for intuition we can take $\delta = 1/100$, $d = 1/100$, and $D = 10$, but we do not compute exact values.

Let V be a *fixed* codimension k subspace of \mathbb{F}_q^n . Then we say that V is *sparse*, *unsaturated*, *semi-saturated*, or *saturated* as follows.

sparse: There is a non-zero $w \perp V$ with $|\text{supp } w| \leq \delta n$. We can directly count these subspaces.

unsaturated: V is not sparse and we have the estimate

$$\max(e^{-d\alpha n}, Dq^{-k}) < |\mathbb{P}(X \in V) - q^{-k}|.$$

We adapt the swapping method from [13] and construct a random vector Y such that $\mathbb{P}(X \in V) \leq (\frac{1}{2} + \frac{1}{D} + o(1))\mathbb{P}(Y \in V)$.

semi-saturated: V is not sparse and we have the estimates

$$e^{-d\alpha n} < |\mathbb{P}(X_k \in V) - q^{-k}| \leq Dq^{-k}.$$

In this range the swapping method does not yield a useful gain; however, we can enumerate semi-saturated V by finding a structured $w \perp V$. Note that for q sufficiently large there are no semi-saturated spaces.

saturated: V is not sparse and we have the estimate

$$|\mathbb{P}(X_k \in V) - q^{-k}| \leq e^{-d\alpha n}.$$

Proposition 2.1 will follow if we can show that W_k represents a saturated subspace with probability $1 - O(e^{-c\alpha n})$ with absolute constants. It therefore suffices to show that W_k is sparse, semi-saturated, or unsaturated with probability $O(e^{-c\alpha n})$.

2.1. Sparse subspaces. We adapt the counting method from [6]. If W_k is sparse, then we can find a non-zero $w \perp W_k$ with $|\text{supp } w| \leq \delta n$. By the union bound,

$$\mathbb{P}(W_k \text{ is sparse}) \leq \sum_{\substack{\sigma \subseteq [n] \\ 1 \leq |\sigma| \leq \delta n}} \mathbb{P}(W_k \perp w \text{ for some } w \text{ with } \text{supp } w = \sigma).$$

Fix σ . It suffices to bound

$$Q_\sigma := \mathbb{P}(W_k \perp w \text{ for some } w \text{ with } \text{supp } w = \sigma) \leq O(e^{-c\alpha n})$$

with the implied constant and $c > 0$ depending on δ . We will choose δ in the proofs for unsaturated and semi-saturated subspaces.

If we have such a perpendicular vector w we can write the matrix equation

$$w^t [X_{\ell+1} \ \cdots \ X_n] = 0.$$

Restricting the product to indices in σ and denoting this reduction by $\tilde{\cdot}$,

$$\tilde{w}^t [\tilde{X}_{\ell+1} \ \cdots \ \tilde{X}_n] = 0.$$

The matrix of reduced columns has size $|\sigma| \times (n - \ell)$ and has rank less than $|\sigma|$, so we conclude that the dimension of the column space is at most $|\sigma| - 1$. There are at most $\binom{n-\ell}{|\sigma|-1}$ possible choices for a set τ of spanning columns; we do not require that they be linearly independent. Regardless of the choice of τ , the remaining columns must be perpendicular to \tilde{w} . Collecting these bounds, we find

$$Q_\sigma \leq \sum_{\substack{\tau \subset [n] \\ |\tau|=|\sigma|-1}} \sup_{\text{supp } w=\sigma} \mathbb{P}(X_t \perp w \text{ for all } t \notin \tau \mid \text{codim } W_k = k)$$

We expect linearly independent vectors to be less likely to lie in a given subspace than average. The next proposition verifies that intuition.

Proposition 2.3. *Let Z_1, \dots, Z_r be non-trivial iid random vectors in \mathbb{F}_q^n . Then we have the bound*

$$\mathbb{P}(Z_1, \dots, Z_r \in V \mid Z_1, \dots, Z_r \text{ are linearly independent}) \leq \mathbb{P}(Z \in V)^r.$$

Proof. Expanding the left hand side with conditional expectation,

$$\prod_{j=1}^r \mathbb{P}(Z_j \in V \mid Z_1, \dots, Z_{j-1} \in V \text{ and } Z_1, \dots, Z_j \text{ are linearly independent})$$

Let $U := \langle Z_1, \dots, Z_{j-1} \rangle \triangleleft V$ denote the span of the exposed vectors. It suffices to show that

$$\frac{\mathbb{P}(Z \in V \setminus U)}{\mathbb{P}(Z \notin U)} \leq \mathbb{P}(Z \in V).$$

In fact,

$$\begin{aligned} \mathbb{P}(Z \in V \setminus U) &= \mathbb{P}(Z \in U) \mathbb{P}(Z \in V \setminus U) + \mathbb{P}(Z \notin U) \mathbb{P}(Z \in V \setminus U) \\ &\leq \mathbb{P}(Z \in U) \mathbb{P}(Z \notin U) + \mathbb{P}(Z \in V \setminus U) \mathbb{P}(Z \notin U) \\ &= (\mathbb{P}(Z \in U) + \mathbb{P}(Z \in V \setminus U)) \mathbb{P}(Z \notin U) \end{aligned}$$

and the proposition follows. \square

Combining terms we get

$$\mathbb{P}(W_k \text{ is sparse}) \leq \sum_{\substack{\sigma \subset [n] \\ 1 \leq |\sigma| \leq \delta n}} \binom{n-\ell}{|\sigma|-1} \sup_{\text{supp } w=\sigma} \mathbb{P}(X \perp w)^{n-\ell-|\sigma|+1}$$

It remains to bound $\mathbb{P}(X \perp w)$ for w with support σ . For this task we can use the following Littlewood-Offord theorem.

Lemma 2.4 (Littlewood-Offord). *Let $X \in \mathbb{F}_q^n$ be a random vector with iid entries taken from an α -dense probability distribution μ . Suppose $w \in \mathbb{F}_q^n$ has at least m non-zero coefficients. Then we have the estimate*

$$\left| \mathbb{P}(X \cdot w = r) - \frac{1}{q} \right| \lesssim \frac{1}{\sqrt{\alpha m}}$$

for all $r \in \mathbb{F}_q$, where the implied constant is absolute.

We only require the estimate for $r = 0$. We will prove Lemma 2.4 in Section 3.

If we combine this with the trivial inequality $\mathbb{P}(X \perp w) \leq 1 - \alpha$ for small $|\sigma|$, we deduce

$$\mathbb{P}(W \text{ is sparse}) \leq O(e^{-c\alpha n})$$

with absolute constants for δ sufficiently small.

2.2. Semi-Saturated subspaces. Let V be a semi-saturated subspace of codimension k . We first claim that we can find a non-zero $\xi \perp V$ that is structured in the following sense.

Proposition 2.5. *For all $\beta > 0$ there is a value of d in the definition of semi-saturated and a subset*

$$R \subseteq \mathbb{F}_q^n, \quad |R| \leq \beta^n q^n$$

such that every semi-saturated V is perpendicular to a non-zero $\xi \in R$.

We will prove Proposition 2.5 in Section 3.

We have therefore found a $\xi \in V^\perp$ that is “structured” in that it lies in an exponentially small subset of \mathbb{F}_q^n . It turns out that this is enough to attain the desired estimate on $\mathbb{P}(W_k \text{ is semi-saturated})$. In fact, we estimate

$$\mathbb{P}(W_k \text{ is semi-saturated} \mid \text{codim } W_k = k) \leq \sum_{\substack{\text{codim } V = k \\ V \text{ is semi-saturated}}} \mathbb{P}(W_k = V \mid \text{codim } W = k).$$

Using Proposition 2.3, we can bound

$$\mathbb{P}(W_k = V \mid \text{codim } W_k = k) \leq \mathbb{P}(X \in V)^{n-k} \leq D^{n-k} q^{-k(n-k)}$$

where the last inequality is from the definition of semi-saturated V . It now suffices to count the number of semi-saturated subspaces.

The subspace V is completely determined by its annihilator V^\perp . We therefore count the number of possible annihilators that meet R . We can choose k generators v_1, \dots, v_k for V^\perp and force $v_1 \in R$; we then divide by the number of ways we could generate the same subspace with different choices for v_2, \dots, v_k . This gives the upper bound

$$\#\{\text{semi-saturated } V\} \lesssim \beta^n q^n \frac{(q^n)^{k-1}}{|V^\perp|^{k-1}} \leq \beta^n q^{nk-k^2+k}.$$

Collecting terms we find

$$\mathbb{P}(W_k \text{ is semi-saturated}) \lesssim D^{n-k} \beta^n q^k$$

If there are any semi-saturated subspaces, we must have the inequality $e^{-d\alpha n} \leq Dq^{-k}$. With fixed D we can choose β and therefore an upper bound for d such that the right hand side converges to zero at an exponential rate.

2.3. Unsaturated subspaces. In [13] it was observed that there is a random vector Y such that if $X \in \mathbb{R}^n$ is a Bernoulli random vector and V is a non-sparse hyperplane, then we can bound

$$\mathbb{P}(X \in V) \leq \left(\frac{1}{2} + o(1)\right) \mathbb{P}(Y \in V).$$

Ignoring difficulties with independence, this suggests the inequality

$$\mathbb{P}(X_{k+1}, \dots, X_n \text{ span } V) \leq c^n \mathbb{P}(Y_{k+1}, \dots, Y_n \text{ span } V)$$

for some $1/2 < c < 1$. Summing over non-saturated subspaces V and using the trivial bound

$$\sum_{\substack{V \text{ unsaturated} \\ \text{codim } V = k}} \mathbb{P}(Y_{k+1}, \dots, Y_n \text{ span } V) \leq 1$$

would complete the argument.

Over the finite field \mathbb{F}_q we cannot quite get the above inequality, but rather an inequality of the form

$$|\mathbb{P}(X \in V) - q^{-k}| \leq \left(\frac{1}{2} + o(1)\right) |\mathbb{P}(Y \in V) - q^{-k}|.$$

This reflects our intuition that Fourier analysis over \mathbb{F}_q controls errors from univormity rather than absolute probabilities. If we want to use this inequality to get an exponential strength gain, then we must require $\mathbb{P}(X \in V) - q^{-k} > Dq^{-k}$ for some $D > 0$. It turns out that this is enough for the argument to work.

Let ν denote a probability distribution to be chosen later. Suppose ν is β -dense for some $\beta > 0$; we will later show that $\beta = \alpha/8$. Let $Y_1, \dots, Y_r \in \mathbb{F}_q^n$ be iid random vectors with iid entries taken from ν and let $Z_1, \dots, Z_s \in \mathbb{F}_q^n$ be iid copies of X . Here r, s are parameters to be chosen later.

We will need control over $\mathbb{P}(X \in V)$ in the sequel. We therefore make the following definition, first given in [13].

Definition 2.6. Let V be a deterministic subspace in \mathbb{F}_q^n . We say that V has combinatorial codimension $d_\pm \in \mathbb{Z}^+/n$ and write $d_\pm(V) = d_\pm$ if

$$(1 - \alpha)^{d_\pm} \leq \mathbb{P}(X \in V) < (1 - \alpha)^{d_\pm - 1/n}$$

Note that the combinatorial codimension of a subspace depends on the choice of α and μ . There are $O(n^2)$ possible combinatorial codimensions, so it suffices to control each separately.

In this section, we will assume that X_{k+1}, \dots, X_n are conditioned to be linearly independent.

Fix an unsaturated subspace V with codimension k and combinatorial codimension d_\pm . Let B_V denote the event

$$B_V := "Y_1, \dots, Y_r, Z_1, \dots, Z_s \text{ are linearly independent in } V."$$

By probabilistic independence we can write

$$\mathbb{P}(W_k = V) = \frac{\mathbb{P}(B_V \wedge W_k = V)}{\mathbb{P}(B_V)}.$$

If X_{k+1}, \dots, X_n span V , we can find $n-k-r-s$ columns that complete $Y_1, \dots, Y_r, Z_1, \dots, Z_s$ to a basis for V . The remaining vectors must also lie in V . We therefore define the event

$$C_V := "X_{k+r+s+1}, \dots, X_n, Y_1, \dots, Y_r, Z_1, \dots, Z_s \text{ span } W"$$

so after relabeling the columns of A ,

$$\mathbb{P}(B_V \wedge W = V) \leq \binom{n-k}{r+s} \mathbb{P}(X_{k+1}, \dots, X_{k+r+s} \in V) \mathbb{P}(C_V)$$

By Proposition 2.3, recalling that our vectors X_{k+1}, \dots, X_n are conditioned to be linearly independent,

$$\mathbb{P}(X_{k+1}, \dots, X_{k+r+s} \in V) \leq \mathbb{P}(X \in V)^{r+s}.$$

Next we consider $\mathbb{P}(B_V)$. We can write by conditional expectation

$$\mathbb{P}(B_V) = \mathbb{P}(B_V \mid Y_1, \dots, Y_r, Z_1, \dots, Z_s \in V) \mathbb{P}(Y \in V)^r \mathbb{P}(Z \in V)^s.$$

We need to control the probability that the vectors $Y_1, \dots, Y_r, Z_1, \dots, Z_s$ are linearly independent. It turns out that Odlyzko's lemma is strong enough for what we need, as long as r and s are not too large and the combinatorial codimension is not too small.

Proposition 2.7. *Let Y_1, \dots, Y_r be iid vectors taken from a β -dense probability distribution ν and let Z_1, \dots, Z_s be iid vectors taken from an α -dense probability distribution μ . Then if V has combinatorial codimension $d_{\pm} \leq O_{\alpha, \beta}(n)$ we have*

$$\mathbb{P}(B_V \mid Y_1, \dots, Y_r, Z_1, \dots, Z_s \in V) \geq \frac{1}{2}.$$

Proof. Define the events

$$F_V := "Y_1, \dots, Y_r, Z_1, \dots, Z_s \in V."$$

and, for convenience,

$$F_V(i) := "F_V \wedge Y_1, \dots, Y_{i-1} \text{ are linearly independent}"$$

$$\tilde{F}_V(j) := "F_V \wedge Y_1, \dots, Y_r, Z_1, \dots, Z_{j-1} \text{ are linearly independent}."$$

Expanding the probability with conditional expectation,

$$\begin{aligned} \mathbb{P}(B_V \mid F_V) &= \prod_{i=1}^r \mathbb{P}(Y_i \notin \langle Y_1, \dots, Y_{i-1} \rangle \mid F_V(i)) \\ &\quad \times \prod_{j=1}^s \mathbb{P}(Z_j \notin \langle Y_1, \dots, Y_r, Z_1, \dots, Z_{j-1} \rangle \mid \tilde{F}_V(j)). \end{aligned}$$

With Lemma 2.2,

$$\mathbb{P}(Y_i \notin \langle Y_1, \dots, Y_{i-1} \rangle \mid F_V(i)) \geq 1 - (1 - \beta)^{n-i+1} (1 - \alpha)^{-d_{\pm}}$$

and

$$\mathbb{P}(Z_j \notin \langle Y_1, \dots, Y_r, Z_1, \dots, Z_{j-1} \rangle \mid \tilde{F}_V(j)) \geq 1 - (1 - \alpha)^{n-r-j+1} (1 - \alpha)^{-d_{\pm}}$$

We therefore have the lower bound

$$\mathbb{P}(B_V \mid F_V) \geq 1 - (1 - \beta)^{n-i+1} (1 - \alpha)^{-d_{\pm}} - (1 - \alpha)^{n-r-j+1} (1 - \alpha)^{-d_{\pm}} \geq 1/2$$

as long as d_{\pm} is sufficiently small and r, s are sufficiently small. \square

Collecting estimates, we have

$$\mathbb{P}(W = V) \lesssim \binom{n-k}{r+s} \frac{\mathbb{P}(X \in V)^r}{\mathbb{P}(Y \in V)^r} \mathbb{P}(C_V)$$

We are now ready to state the key lemma to compare the random vectors X and Y .

Lemma 2.8 (Swapping). *There is a β -dense probability distribution ν on \mathbb{F}_q with $\beta = \alpha/8$ such that, if $Y \in \mathbb{F}_q^n$ is a random vector with iid coefficients distributed according to ν , then*

$$|\mathbb{P}(X \in V) - q^{-1}| \leq \left(\frac{1}{2} + o(1) \right) |\mathbb{P}(Y \in V) - q^{-1}|.$$

If V is unsaturated, then as an immediately corollary we have

$$\mathbb{P}(X \in V) \leq \left(\frac{1}{2} + \frac{1}{D} + o(1) \right) \mathbb{P}(Y \in V)$$

We will prove this lemma in Section 3. With this estimate, we can sum over all subspaces of codimension k and combinatorial codimension e . Since a set of vectors can span at most one subspace, the events C_V for V varying are disjoint and we can conclude

$$\sum_{\substack{V: \text{codim } V = k \\ d_{\pm}(V) = d_{\pm}}} \mathbb{P}(W = V) \lesssim \binom{n-k}{r+s} 2^{-r} = O(e^{-cn}).$$

Here we picked $r = \delta_1 n$, $s = n - k - r - \delta_2 n$. \square

3. LITTLEWOOD-OFFORD THEOREMS

We now come to the heart of the argument: proving the three Littlewood-Offord type lemmas used in the preceding section.

We briefly review some theory from additive combinatorics. For more discussion, see [12].

The following cosine inequality is elementary.

Lemma 3.1. *For all positive integers k and for any $\beta_1, \dots, \beta_k \in \mathbb{R}$ we have the inequality*

$$\cos(\beta_1 + \dots + \beta_k) \geq k \sum_{\ell=1}^k \cos \beta_{\ell} - k^2 + 1.$$

Proof. We can assume that $-\pi/2 \leq \beta_{\ell} \leq \pi/2$ for all ℓ , as otherwise the inequality is trivial. On this interval \cos is concave, so we have the inequality

$$k^{-1} \sum_{\ell=1}^k \cos \beta_{\ell} \leq \cos \left(\frac{\beta_1 + \dots + \beta_k}{k} \right).$$

It suffices to show that

$$\cos(\beta/k) \leq k^{-2} \cos \beta + 1 - k^{-2}$$

for all $\beta \in \mathbb{R}$, but this is immediate from the power series. \square

Let μ be a probability measure on the finite field \mathbb{F}_q . We need estimates on the Fourier transform

$$\widehat{\mu}(\psi) := \sum_{t \in \mathbb{F}_q} \mu(t) \psi(t).$$

Recall that $\mathbb{F}_q \cong \widehat{\mathbb{F}}_q$ via the isomorphism that sends $t \in \mathbb{F}_q$ to the character $x \mapsto e_p(\text{Tr}(tx))$, where $\text{Tr} : F_{p^f} \rightarrow F_p$ is the field trace. We define the additive spectrum $\text{Spec}_{1-\epsilon} \mu$ to be the set

$$\text{Spec}_{1-\epsilon} \mu := \{\psi \in \widehat{\mathbb{F}}_q \mid |\widehat{\mu}(\psi)| \geq 1 - \epsilon\}.$$

For ϵ small, we can find additive structure in $\text{Spec}_{1-\epsilon} \mu$. The next lemma makes this explicit; see Lemma 4.37 in [12].

Lemma 3.2. *For $\epsilon_1, \dots, \epsilon_k < 1$ we have the sum-set inclusion*

$$\text{Spec}_{1-\epsilon_1} \mu + \dots + \text{Spec}_{1-\epsilon_k} \mu \subseteq \text{Spec}_{1-k(\epsilon_1+\dots+\epsilon_k)} \mu.$$

Proof. Let $\psi_\ell \in \text{Spec}_{1-\epsilon_\ell} \mu$ for each ℓ . We write $\psi_\ell(t) = e(\text{Tr}(s_\ell t)/p)$ for appropriate s_ℓ . We can find $\theta_\ell \in \mathbb{R}/\mathbb{Z}$ so that

$$\text{Re} \sum_{t \in \mathbb{F}_q} \mu(t) e(\text{Tr}(s_\ell t)/p + \theta_\ell) \geq 1 - \epsilon_\ell.$$

Summing, we derive

$$\begin{aligned} \text{Re} \sum_{t \in \mathbb{F}_q} \mu(t) (ke(\text{Tr}(s_1 t)/p + \theta_1) + \cdots + ke(\text{Tr}(s_k t)/p + \theta_k) - k^2 + 1) \\ \geq 1 - k(\epsilon_1 + \cdots + \epsilon_k). \end{aligned}$$

The result now follows from Lemma 3.1. \square

A subset $A \subseteq Z$ of an abelian group induces a symmetry subgroup of Z given by

$$\text{Sym } A := \{h \in Z \mid h + A = A\}$$

Clearly A can be decomposed into the union of cosets of $\text{Sym } A$.

We need to bound sumsets from below. For $q = p$ the estimate we need is the Cauchy-Davenport inequality: any $A, B \subseteq \mathbb{F}_p$ satisfy $|A+B| \geq \min(|A|+|B|-1, p)$. The next lemma generalizes the Cauchy-Davenport inequality to non-cyclic groups; see Theorem 5.5 in [12] for a proof.

Lemma 3.3 (Kneser's Theorem). *Let $A, B \subseteq Z$ be finite subsets of an abelian group Z . We have the lower bound*

$$|A+B| + |\text{Sym}(A+B)| \geq |A| + |B|.$$

Since $\text{Sym}(A_1 + \cdots + A_k)$ is increasing in k , we get the following iterated version.

Corollary 3.4. *Let $A_1, \dots, A_k \subseteq Z$ be finite subsets of an abelian group Z . We have the lower bound*

$$|A_1 + \cdots + A_k| + (k-1)|\text{Sym}(A_1 + \cdots + A_k)| \geq |A_1| + \cdots + |A_k|.$$

3.1. The Classical Littlewood-Offord Estimate. We start by bounding the concentration probability $\mathbb{P}(X \cdot w = r)$ for arbitrary $r \in \mathbb{F}_q$, $X \in \mathbb{F}_q^n$ a random vector with iid entries taken from an α -dense probability measure μ , and $w \in \mathbb{F}_q^n$ a vector with at least m non-zero entries.

Proof of Lemma 2.4. Let ξ_1, \dots, ξ_n denote the entries of X . We can decompose the concentration probability into its Fourier transform,

$$\mathbb{P}(X \cdot w = r) = q^{-1} + q^{-1} \sum_{t \in \mathbb{F}_q \setminus \{0\}} e_p(\text{Tr}(-rt)) \prod_{\ell=1}^n \mathbb{E} e_p(\text{Tr}(\xi_\ell w_\ell t)).$$

By the triangle inequality,

$$|\mathbb{P}(X \cdot w = r) - q^{-1}| \leq q^{-1} \sum_{t \in \mathbb{F}_q \setminus \{0\}} \prod_{\ell=1}^n |\mathbb{E} e_p(\text{Tr}(\xi_\ell w_\ell t))|$$

Note that $\mathbb{E} e_p(\text{Tr}(\xi_\ell w_\ell t)) = \widehat{\mu}(w_\ell t)$.

We define $\psi(t) := 1 - |\widehat{\mu}(t)|^2$ so that, with the inequality $|\theta| \leq \exp(-\frac{1}{2}(1 - \theta^2))$, we have

$$|\mathbb{P}(X \cdot w = r) - q^{-1}| \leq q^{-1} \sum_{t \in \mathbb{F}_q \setminus \{0\}} \exp\left(-\frac{1}{2} \sum_{\ell=1}^n \psi(w_\ell t)\right)$$

Put $f(t) := \sum_{\ell} \psi(w_{\ell} t)$. We can decompose the sum into level sets,

$$|\mathbb{P}(X \cdot w = r) - q^{-1}| \leq \frac{1}{2} \int_0^{\infty} q^{-1} |\{t \neq 0 \mid f(t) \leq v\}| e^{-v/2} dv.$$

Let $T(v) := \{t \mid f(t) \leq v\}$ and $T'(v) := T(v) \setminus \{0\}$.

We claim the following sum-set inequality; see [4] for the torsion-free case.

Proposition 3.5. *For any $v > 0$, we have the inclusion*

$$T(v) + \cdots + T(v) \subseteq T(k^2 v)$$

where there are k terms in the sum.

Proof. We first observe that for any $\beta_1, \dots, \beta_k \in \mathbb{F}_q$, we have the inequality

$$\psi(\beta_1 + \cdots + \beta_k) \leq k(\psi(\beta_1) + \cdots + \psi(\beta_k)).$$

In fact, we can rewrite this equation as

$$\begin{aligned} 1 - \sum_{a, b \in \mathbb{F}_q} \mu(a) \mu(-b) \cos\left(\frac{2\pi}{p} \text{Tr}((a+b)(\beta_1 + \cdots + \beta_k))\right) \\ \leq k^2 - k \sum_{j=1}^k \sum_{a, b \in \mathbb{F}_q} \mu(a) \mu(-b) \cos\left(\frac{2\pi}{p} \text{Tr}((a+b)\beta_j)\right) \end{aligned}$$

which follows from Lemma 3.1.

Suppose t_1, \dots, t_k satisfy $f(t_k) \leq v$. Then we have

$$f(t_1 + \cdots + t_k) = \sum_{\ell=1}^n \psi(w_{\ell} t_1 + \cdots + w_{\ell} t_k) \leq k \sum_{j=1}^k \sum_{\ell=1}^n \psi(w_{\ell} t_j) \leq k^2 v$$

as required. \square

By Corollary 3.4 we deduce

$$k|T(v)| \leq |T(k^2 v)| + (k-1)|\text{Sym}(T(v) + \cdots + T(v))|.$$

This inequality is effective as long as $|\text{Sym}(T(v) + \cdots + T(v))| = 1$. If $\text{Sym}(T(v) + \cdots + T(v)) \neq \{0\}$, then because $T(v) + \cdots + T(v) \subseteq T(k^2 v)$ we can find a non-trivial additive subgroup $H \triangleleft \mathbb{F}_q$ contained in the set $T(k^2 v)$. It therefore suffices to choose k such that $T(k^2 v)$ contains no non-trivial additive subgroups.

Fix H ; we will find a $t \in H$ where f is large. Averaging f over the subgroup,

$$|H|^{-1} \sum_{t \in H} f(t) = \sum_{\ell=1}^n |H|^{-1} \sum_{t \in H} \psi(w_{\ell} t) = \sum_{\ell=1}^n |H|^{-1} \sum_{t \in H} (1 - |\widehat{\mu}(w_{\ell} t)|^2).$$

By the inverse Fourier transform and the α -density of μ ,

$$|H|^{-1} \sum_{t \in H} |\widehat{\mu}(w_{\ell} t)|^2 = \sum_{\xi, \zeta \in \mathbb{F}_q} \mu(\xi) \mu(\zeta) 1_{H^{\perp}}(w_{\ell}(\xi - \zeta)) \leq 1 - \alpha$$

Since at least m of the coefficients w_{ℓ} are non-zero,

$$|H|^{-1} \sum_{t \in H} f(t) \geq \alpha m.$$

By the pigeonhole principle, there must be a $t \in H$ with $f(t) \geq \alpha m$.

We therefore conclude that

$$|T'(v)| \lesssim \sqrt{\frac{v}{\alpha m}} |T'(\alpha m)|$$

for all $v \leq \alpha m$. Inserting this inequality into the level set estimate gives the bound

$$|\mathbb{P}(X \cdot w \equiv r) - q^{-1}| \lesssim \frac{1}{\sqrt{\alpha m}} \int_0^\infty \sqrt{v} e^{-v} dv + e^{-\alpha m/2}$$

as required. \square

3.2. The Inverse Theorem. We can find our structured perpendicular vector ξ with the pigeonhole principle.

Proof of Proposition 2.5. Let $k := \text{codim } V$. We take Fourier transforms to find

$$|\mathbb{P}(X \in V) - q^{-k}| \leq q^{-k} \sum_{\zeta \in V^\perp \setminus \{0\}} \prod_{\ell=1}^n |\widehat{\mu}(\zeta_\ell)|$$

By the pigeonhole principle, we can bound this above by

$$\prod_{\ell=1}^n |\widehat{\mu}(\xi_\ell)|$$

for some fixed $\xi \in V^\perp \setminus \{0\}$. Since V is not sparse, $|\text{supp } \xi| \geq \delta n$.

Because V is semi-saturated, we get the lower bound

$$e^{-dn} \leq \prod_{\ell=1}^n |\widehat{\mu}(\xi_\ell)|.$$

With the estimate $|\theta| \leq \exp(-\frac{1}{2}(1 - \theta^2))$ we can take logarithms to find

$$\sum_{\ell=1}^n 1 - |\widehat{\mu}(\xi_\ell)|^2 \leq dn.$$

Let $\epsilon = 5d$. We can choose $\sigma \subseteq [n]$ with $|\sigma| \geq 0.9n$ such that

$$\xi_\ell \in \text{Spec}_{1-\epsilon} \mu$$

for all $\ell \in \sigma$.

It suffices to find an absolute $\eta > 0$ such that $|\text{Spec}_{1-\eta} \mu| \leq \beta q$. We observe that there is a value $\gamma > 0$ such that $\text{Spec}_{1-\gamma} \mu$ does not contain any non-trivial additive subgroups $H \triangleleft \mathbb{F}_q^n$. In fact, by Markov's inequality and Fourier inversion,

$$(1 - \gamma)^2 \#H \cap \text{Spec}_{1-\gamma} \mu \leq \sum_{t \in H} |\widehat{\mu}(t)|^2 \leq |H|(1 - \alpha).$$

We then choose $\gamma = \alpha/2$.

We can now use Corollary 3.4 to show that for any $k \geq 1$,

$$|\text{Spec}_{1-\gamma} \mu \setminus \{0\}| \geq k |\text{Spec}_{1-k^{-2}\gamma} \mu \setminus \{0\}|$$

so we pick $k = \beta^{-1}$ and let $\delta = \beta^2 \gamma = \beta^2 \alpha/2$. We then deduce that $\beta = \sqrt{c/(5\alpha)}$. \square

3.3. The Swapping Lemma. Let μ be an α -dense probability distribution and V an unsaturated subspace of codimension k . We want to find ν depending only on μ such that, if Y is a random vector with iid entries taken from ν , we have the inequality

$$|\mathbb{P}(X \in V) - q^{-k}| \leq \left(\frac{1}{2} + o(1)\right) |\mathbb{P}(Y \in V) - q^{-k}|$$

Let us postpone the definition of ν and define functions $f, g : \mathbb{F}_q \rightarrow \mathbb{R}^+$ to be

$$f(t) = \prod_{\ell=1}^n |\hat{\mu}(w_\ell t)|$$

$$g(t) = \prod_{\ell=1}^n \hat{\nu}(w_\ell t).$$

The lemma would follow immediately if we had $\hat{\nu} \geq 0$ and we could establish

$$\sum_{t \in V^\perp \setminus \{0\}} f(t) \leq \left(\frac{1}{2} + o(1)\right) \sum_{t \in V^\perp \setminus \{0\}} g(t).$$

Let $F(u) = \{t \mid f(t) \geq u\}$ and $G(u) = \{t \mid g(t) \geq u\}$ denote level sets. We define ν so that the level sets $G(u)$ control the additive structure of $F(u)$.

Proposition 3.6. *There is a probability distribution $\nu : \mathbb{F}_q \rightarrow [0, 1]$ depending on μ and α with the following properties.*

- (1) *For all $0 < u < 1$ we have the sumset inclusion $F(u) + F(u) \subseteq G(u)$.*
- (2) *For all $t \in V^\perp$, $f(t) \leq g(t)^4$.*
- (3) *$\hat{\nu}(t) \geq 0$ for all $t \in \hat{\mathbb{F}}_q$.*
- (4) *ν is β -dense for $\beta = \alpha/8$.*

We will prove Proposition 3.6 in a moment. First we will show how to use Proposition 3.6 to prove Lemma 2.8.

Proof of Lemma 2.8. Let $\epsilon > 0$ be determined later. We decompose the sum of f into the domains where $f \leq \epsilon$ and $f > \epsilon$,

$$\sum_{t \in V^\perp \setminus \{0\}} f(t) \leq \sum_{\substack{t \in V^\perp \setminus \{0\} \\ f(t) \leq \epsilon}} f(t) + \sum_{\substack{t \in V^\perp \setminus \{0\} \\ f(t) > \epsilon}} f(t).$$

We can control the domain where $f \leq \epsilon$ using the inequality $f(t) \leq g(t)^4$. Namely,

$$\sum_{\substack{t \in V^\perp \setminus \{0\} \\ f(t) \leq \epsilon}} f(t) \leq \epsilon^{3/4} \sum_{t \in V^\perp \setminus \{0\}} g(t).$$

Therefore if we can set $\epsilon = o(1)$ as $n \rightarrow \infty$ this part is complete.

We write the sum over the domain where $f > \epsilon$ into level sets,

$$\sum_{\substack{t \in V^\perp \setminus \{0\} \\ f(t) > \epsilon}} f(t) = \int_\epsilon^\infty |F'(u)| du + \epsilon |F'(\epsilon)|$$

Here we let $F'(u) := F(u) \setminus \{0\}$ and similarly define $G'(u) := G(u) \setminus \{0\}$.

From the sumset inequality $F(u) + F(u) \subseteq G(u)$ and Kneser's inequality,

$$2|F(u)| \leq |\text{Sym}(F(u) + F(u))| + |G(u)|$$

We would like to pick $\epsilon = o(1)$ such that $|\text{Sym}(F(u) + F(u))| = 1$. Since $F(u) + F(u) \subseteq G(u)$ and $G(u)$ is increasing in u , we require that every non-trivial additive subgroup $H \triangleleft V^\perp$ contain a non-zero element $t \notin G(\epsilon)$.

Fix $H \triangleleft V^\perp$. We can clearly assume that $H \cong \mathbb{Z}/p\mathbb{Z}$; pick $w \in V^\perp$ that generates H . Since V is unsaturated, we know that w contains at least δn non-zero entries.

Define the function

$$h(t) := \sum_{\ell=1}^n 1 - \tilde{\nu}(t_\ell)^2.$$

for $t \in H$. Averaging h over H , we can argue as in the proof of Lemma 2.4 to find

$$|H|^{-1} \sum_{t \in H} h(t) \geq \beta \delta n.$$

Note that we need ν to be β -dense. By the pigeonhole principle we can find a (non-zero) $t \in H$ with $h(t) \geq \beta \delta n$. We then deduce that

$$g(t) \leq \exp(-\frac{1}{2}h(t)) \leq \exp(-\frac{1}{2}\beta \delta n)$$

so we set $\epsilon = \exp(-\frac{1}{2}\beta \delta n)$. For every $u \geq \epsilon$ we now have

$$2|F'(u)| \leq |G'(u)|,$$

so returning to our integral of level sets we find

$$\int_{\epsilon}^{\infty} |F'(u)| du + \epsilon |F'(\epsilon)| \leq \frac{1}{2} \int_0^{\infty} |G'(u)| du.$$

The lemma now follows. \square

Proof of Proposition 3.6. Let $\gamma = 1/8$ be a parameter and define

$$\nu(t) := \begin{cases} \gamma \mu * \mu^-(t), & t \neq 0 \\ 1 - \sum_{s \neq 0} \nu(s), & t = 0. \end{cases}$$

Clearly ν is a probability measure if $0 < \gamma < 1$. We also have $\hat{\nu} > 1 - 2\gamma$. Let $\beta = \gamma\alpha$. It is easy to see that ν is β -dense: for $H \triangleleft \mathbb{F}_q$ additive we have

$$\nu(H) = 1 - \sum_{t \notin H} \gamma \mu * \mu^-(t) \leq 1 - \gamma\alpha$$

and for any $x \notin H$ we have

$$\nu(x + H) = \sum_{t \in x + H} \gamma \mu * \mu^-(t) \leq \gamma(1 - \alpha) \leq 1 - \gamma\alpha.$$

as desired.

The Fourier transform of ν is given by

$$\hat{\nu}(\xi) = 1 - \gamma + \gamma |\hat{\mu}(\xi)|^2.$$

We would next like to show that $F(u) + F(u) \subseteq G(u)$ for all $0 < u < 1$. It suffices to show, for all $\theta, \psi \in \widehat{\mathbb{F}}_q$,

$$|\hat{\mu}(\theta)\hat{\mu}(\psi)| \leq \hat{\nu}(\theta + \psi)^2.$$

We will consider two cases.

(1) Suppose $|\hat{\mu}(\theta)| < 1 - 4\gamma$ or $|\hat{\mu}(\psi)| < 1 - 4\gamma$. Then

$$|\hat{\mu}(\theta)\hat{\mu}(\psi)| < 1 - 4\gamma < (1 - 2\gamma)^2 < \hat{\nu}^2(\theta + \psi).$$

- (2) Now suppose that $|\hat{\mu}(\theta)|, |\hat{\mu}(\psi)| \geq 1 - 4\gamma$. Define $\theta_1 = 1 - |\hat{\mu}(\theta)|$ and $\theta_2 = 1 - |\hat{\mu}(\psi)|$. By Lemma 3.2, we know that $|\hat{\mu}(\theta + \psi)|^2 \geq 1 - 2(\theta_1 + \theta_2)$.

We have the inequality

$$\hat{\nu}(\theta + \psi) = 1 - \gamma + \gamma|\hat{\mu}(\theta + \psi)|^2 \geq 1 - 4\gamma(\theta_1 + \theta_2)$$

Since we have $\gamma = 1/8$, we conclude that

$$\hat{\nu}(\theta + \psi)^2 \geq |\hat{\mu}(\theta)\hat{\mu}(\psi)|$$

as required.

It remains to show that $|\hat{\mu}(\theta)| \leq \hat{\nu}(\theta)^4$ for all θ . By the geometric-arithmetic mean inequality,

$$(|\hat{\mu}(\theta)|^2 \cdot 1^7)^{1/8} \leq \frac{1}{8}(|\hat{\mu}(\theta)|^2 + 7) = \hat{\nu}(\theta)$$

as required. \square

4. PROBABILITY DISTRIBUTION OF THE DETERMINANT

We will now indicate how to modify the proof of Theorem 1.2 to prove Theorem 1.3.

Again let X_1, \dots, X_n denote the columns of A . We begin by revealing all but the first column of the matrix. If we abbreviate $W := \langle X_2, \dots, X_n \rangle$ then we find

$$\mathbb{P}(\det M = t) = \mathbb{P}(\det M = t \mid \text{codim } W = 1) \mathbb{P}(\text{codim } W = 1)$$

We now use Proposition 2.1 to control the last $n - 1$ vectors,

$$\begin{aligned} \mathbb{P}(\text{codim } W = 1) &= \prod_{k=2}^n \mathbb{P}(X_k \notin \langle X_{k+1}, \dots, X_n \rangle \mid \text{codim } \langle X_{k+1}, \dots, X_n \rangle = k) \\ &= \prod_{k=2}^{\infty} (1 - q^{-k}) + O(e^{-cn}). \end{aligned}$$

Pick $w \perp W$ such that $\det A = X_1 \cdot w$; namely, w is the first row of the adjugate of A .

We can classify the possible hyperplanes V that W can represent. These are similar to the definitions made in Section 2, but the definition of semi-saturated has been expanded.

sparse: We have $|\text{supp } w| \leq \delta n$. Note that this is well-defined independent of the choice of $w \perp W$.

unsaturated: V is not sparse and either

$$\max(e^{-d\alpha n}, Dq^{-1}) \leq |\mathbb{P}(X \in V) - q^{-1}|$$

or

$$Dq^{-1} \leq \mathbb{P}(X \in V) \leq e^{-d\alpha n} \leq \mathbb{P}(X \cdot w = t)$$

for some $t \in \mathbb{F}_q$.

semi-saturated: V is not sparse,

$$|\mathbb{P}(X \in V) - q^{-1}| < Dq^{-1}$$

and there is a $t \in \mathbb{F}_q$ with

$$e^{-d\alpha n} < |\mathbb{P}(X \cdot w = t) - q^{-1}|.$$

We can control these with a modified the inverse theorem.

saturated: V is not sparse and

$$|\mathbb{P}(X \cdot w = t) - q^{-1}| \leq e^{-d\alpha n}$$

for all $t \in \mathbb{F}_q$.

We will now show that W represents sparse, semi-saturated, and unsaturated subspaces with probability $O(e^{-c\alpha n})$.

4.1. Sparse subspaces. The argument in Section 2.1 shows that these occur with probability $O(e^{-c\alpha n})$.

4.2. Unsaturated subspaces. Since $\mathbb{P}(X \cdot w = t) \leq \mathbb{P}(Y \in V)$, we see that regardless of which set of inequalities hold, we have

$$Dq^{-1} \leq \mathbb{P}(X \in V)$$

and

$$e^{-d\alpha n} \leq \mathbb{P}(Y \in V).$$

Therefore the argument from Section 2.3 applies, so that unsaturated subspaces appear with probability $O(e^{-c\alpha n})$.

4.3. Semi-saturated subspaces. For all $t \in \mathbb{F}_p$ we can calculate

$$\mathbb{P}(X \cdot w = t) = q^{-1} \sum_{\xi \in \mathbb{Z}/(p)} e_p(-\text{Tr}(t\xi)) \prod_{\ell=1}^n \mathbb{E} e_p(\text{Tr}(\psi_\ell w_\ell \xi))$$

Rearranging and applying the triangle inequality,

$$|\mathbb{P}(X \cdot w = t) - q^{-1}| \leq q^{-1} \sum_{\xi \in \mathbb{F}_q \setminus \{0\}} \prod_{\ell=1}^n |\cos(2\pi w_\ell \xi)|$$

The argument can now be completed as in Theorem 1.2. \square

5. ACKNOWLEDGMENTS

The author thanks Terence Tao for guidance and helpful conversation.

REFERENCES

- [1] Jean Bourgain, Van H. Vu, and Philip Matchett Wood. On the singularity probability of discrete random matrices. *J. Funct. Anal.*, 258(2):559–603, 2010.
- [2] Leonard S. Charlap, Howard D. Rees, and David P. Robbins. The asymptotic probability that a random biased matrix is invertible. *Discrete Math.*, 82(2):153–163, 1990.
- [3] P. Erdős. On a lemma of Littlewood and Offord. *Bull. Amer. Math. Soc.*, 51:898–902, 1945.
- [4] G. Halász. Estimates for the concentration function of combinatorial number theory and probability. *Period. Math. Hungar.*, 8(3-4):197–211, 1977.
- [5] Jeff Kahn and János Komlós. Singularity probabilities for random matrices over finite fields. *Combin. Probab. Comput.*, 10(2):137–157, 2001.
- [6] Jeff Kahn, János Komlós, and Endre Szemerédi. On the probability that a random ± 1 -matrix is singular. *J. Amer. Math. Soc.*, 8(1):223–240, 1995.
- [7] J. Komlós. On the determinant of $(0, 1)$ matrices. *Studia Sci. Math. Hungar.*, 2:7–21, 1967.
- [8] J. Komlós. On the determinant of random matrices. *Studia Sci. Math. Hungar.*, 3:387–399, 1968.
- [9] J. E. Littlewood and A. C. Offord. On the number of real roots of a random algebraic equation. III. *Rec. Math. [Mat. Sbornik] N.S.*, 12(54):277–286, 1943.
- [10] Kenneth Maples. Cokernels of random matrices satisfy the Cohen-Lenstra heuristics. *Submitted*.

- [11] A. M. Odlyzko. On subspaces spanned by random selections of ± 1 vectors. *J. Combin. Theory Ser. A*, 47(1):124–133, 1988.
- [12] Terence Tao and Van Vu. *Additive combinatorics*, volume 105 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.
- [13] Terence Tao and Van Vu. On random ± 1 matrices: singularity and determinant. *Random Structures Algorithms*, 28(1):1–23, 2006.
- [14] Terence Tao and Van Vu. On the singularity probability of random Bernoulli matrices. *J. Amer. Math. Soc.*, 20(3):603–628 (electronic), 2007.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CA 90024
E-mail address: `maples@math.ucla.edu`